# REPRESENTABILITY IN SOME SYSTEMS OF SECOND ORDER ARITHMETIC

BY
P. ACZEL*

ABSTRACT

We answer two questions posed in a recent paper by H. B. Enderton by giving characterizations of the sets of integers weakly and strongly representable in a system of second order arithmetic with an infinity rule of inference. The results generalize to each of a family of such systems.

The main aim of this paper is to give characterisations of the sets of integers weakly and strongly representable in a system $(A_{\mathscr{A}})$ of second order arithmetic. This answers questions posed in [1] where this system was introduced. $(A_{\mathscr{A}})$ is obtained from full second order Peano arithmetic $(A)$ (see [1], [3]) by adding the following infinitary rule of inference:

$\mathscr{A}$-rule: For any function $\alpha$, from $\phi(\bar{\alpha}(n))$ for each $n$, infer $\exists v \forall x \phi(\bar{v}(x))$.

We shall give two characterisations of the weakly and strongly $\mathscr{A}$-representable sets. The first is in terms of the class of monotone $\Sigma^1_1$, inductive definitions, while the second is in terms of recursion in the functional $\boldsymbol{E}_1^{\#}$ where, for partial functions $f$

$$\boldsymbol{E}_1^{\#}(f) \simeq \begin{cases} 0 \text{ if } \forall\alpha\exists n f(\bar{\alpha}(n)) = 0 \\ 1 \text{ if } \exists\alpha\forall n f(\bar{\alpha}(n)) > 0 \end{cases}$$

Note that the functional $\boldsymbol{E}_1$ introduced by Tugué is just the restriction of $\boldsymbol{E}_1^{\#}$ to total functions. We shall see that recursion in $\boldsymbol{E}_1$ is very different to recursion in $\boldsymbol{E}_1^{\#}$. We show that if the $\mathscr{A}$-rule is suitably weakened we can define a theory $(A_{w-\mathscr{A}})$ such that representability in this system may be characterised in terms of recursion in $\boldsymbol{E}_1$.

* This paper was written while the author held a Science Research Council fellowship.
Received September 1, 1969.

In §1 we define the $\Sigma_1^1$ and $\mathscr{A}$-inductive definitions and use them to characterise the weakly $\mathscr{A}$-representable sets. In §2 we give schemes sufficient to define the class of functionals partial recursive in any given consistent functional, and state the basic properties of this notion. In §3 we give the characterisations of weak and strong $\mathscr{A}$-representability in terms of recursion in $E_1^\#$. §4 contains a result implying that $E_1^\#$ is much more powerful than $E_1$. The system $(A_{w-\mathscr{A}})$ with the weakened $\mathscr{A}$-rule is introduced in §5 and we characterise weak and strong representability in this system in terms of recursion in $E_1$. Finally in §6 we give definitions and state results that generalise the previous results to a large class of systems of second order arithmetic.

We wish to thank Peter Hinman for informing us of his contribution to proposition $C$ in §2 below which enabled us to complete our characterisation of the strongly $\mathscr{A}$-representable sets.

**Preliminaries.** On the whole we follow the notation of [3] and [1]. For convenience we shall allow set variables as well as function variables in the system $(A)$. Thus atomic formulae of the form $\sigma \in X$ are allowed where $\sigma$ is a term and $X$ is a set variable, and $(\forall X)\phi$, $(\exists X)\phi$ are formulae whenever $\phi$ is. We shall need to add the appropriate axioms for these quantifiers and the comprehension schema $\exists X \forall \alpha[\phi \leftrightarrow x \in X]$ where $X$ is not free in $\phi$. We assume that $(A)$ is axiomatised so that modus ponens is the only rule of inference. We recall that in any system $T$ containing numerals $\boldsymbol{n}$ for integers $n$, a set $A \subseteq \omega$ is weakly $T$-represented by the formula $\phi(v)$ if $A = \{n \in \omega \mid \vdash_T \phi(\boldsymbol{n})\}$. $A$ is weakly (strongly) $T$-representable if there is a formula $\phi(v)$ such that $A$ is weakly $T$-represented by $\phi(v)$ (and also $\omega - A$ is weakly $T$-represented by $\neg\phi(v)$). If $\sigma = \sigma(v_1, \cdots, v_n)$ is a term in the free variables $v_1, \cdots, v_n$ we shall call $\sigma$ a *representing term* for a function $f$ if

$$f(k_1, \cdots, k_n) = m \Rightarrow \vdash_{(A)} \sigma(\boldsymbol{k}_1, \cdots, \boldsymbol{k}_n) = \boldsymbol{m}$$

Note that every recursive function $f$ will have a representing term which we shall write as $\boldsymbol{f}(v_1, \cdots, v_n)$. If $\sigma$ is a representing term with no free variables then there is an integer $n$ such that $\vdash_{(A)} \sigma = \boldsymbol{n}$ If $\sigma, \pi$ are two such terms then $\vdash_{(A)} \sigma = \pi$ or $\vdash_{(A)} \neg(\sigma = \pi)$.

Sometimes it is convenient to extend the language by adding set constants $\boldsymbol{A}$ for $A \subseteq \omega$. Given a sentence $\phi$ in this extended language the truth value of $\phi$ is obtained by interpreting $\phi$ in the standard model of second order arithmetic in the obvious way.

We shall make use of a fixed primitive recursive pairing function $\lambda x, y(x, y)$

with primitive recursive projection functions $\pi$ and $\delta$ We shall write $(x,y,z)$ for $(x,(y,z))$. $\ulcorner\phi\urcorner$ will denote the gödel number of the formula $\phi$ in some standard numbering of formulae. Sometimes we shall not distinguish between a set of formulae and the set of gödel numbers of elements of the sets. Finally we recall from [1] that as well as being closed under the $\mathscr{A}$-rule the $\mathscr{A}$-theorems are closed under the $\omega$-rule:

$$\forall n \vdash_{\mathscr{A}} \phi(\boldsymbol{n}) \Rightarrow \vdash_{\mathscr{A}} \forall x \phi(x),$$

and that this implies closure under the dual $\mathscr{A}$-rule:

$$\forall \alpha \exists n \vdash_{\mathscr{A}} \phi(\bar{\boldsymbol{\alpha}}(\boldsymbol{n})) \Rightarrow \vdash_{\mathscr{A}} \forall v \exists x \phi(\bar{v}(x)).$$

§1. **Monotone $\Sigma_1^1$ inductive definitions.** By an *inductive definition* we shall mean a mapping $\Gamma: P(\omega) \to P(w)$ where $P(\omega) = \{A \mid A \subseteq \omega\}$. Given $\Gamma$ we may define sets of integers $\Gamma^\lambda$ by transfinite recursion on the ordinal $\lambda$; $\Gamma^\lambda = \cup \{\Gamma(\Gamma^\mu) \mid \mu < \lambda\}$. Let $|\Gamma|$ be the least ordinal $\lambda$ such that $\Gamma^{\lambda+1} = \Gamma^\lambda$. Then $|\Gamma|$ exists and is countable. $\Gamma^\infty = \Gamma^{|\Gamma|}$ is the *set inductively defined by* $\Gamma$. We shall only be concerned with *monotone* $\Gamma$ i.e. if $A \subseteq B$ then $\Gamma(A) \subseteq \Gamma(B)$. For such $\Gamma$, $\Gamma(\Gamma^\infty) = \Gamma^\infty$ and $\Gamma^\infty$ may be characterised by the following two properties:

(1) $$\Gamma(\Gamma^\infty) \subseteq \Gamma^\infty$$

(2) $$\Gamma(A) \subseteq A \Rightarrow \Gamma^\infty \subseteq A$$

Hence $\Gamma^\infty$ may be defined by

(3) $$\Gamma^\infty = \cap \{A \subseteq \omega \mid \Gamma(A) \subseteq A\}$$

An inductive definition $\Gamma$ is $\Sigma_1^1$ if there is a $\Sigma_1^1$ predicate $R(\alpha, n)$ such that

$$n \in \Gamma(\{n \in \omega \mid \alpha(n) = 0\}) \Leftrightarrow R(\alpha, n)$$

LEMMA 1.1. *The set of $\mathscr{A}$-theorems is definable by a monotone $\Sigma_1^1$ inductive definition.*

PROOF. Let $Ax$ be the set of axioms of $(A_{\mathscr{A}})$ and let $E$ be the set of formulae of the form $\exists v \forall n \phi(\bar{v}(n))$. Then $Ax.$ and $E$ are recursive sets Let $I$ and $Sb.$ be recursive functions such that

$$I(\ulcorner\phi\urcorner, \ulcorner\psi\urcorner) = \ulcorner\phi \to \psi\urcorner$$

for formulae $\phi$, $\psi$ and

$$Sb.(\ulcorner\exists v \forall n \phi(\bar{v}(n))\urcorner, x) = \ulcorner\phi(\boldsymbol{x})\urcorner$$

for formulae $\phi(v)$ and integers $x$.

The set of $\mathscr{A}$-theorems is definable by the monotone $\Sigma_1^1$ inductive definition $\Gamma$ where

$$\Gamma(A) = Ax. \cup \{a \in \omega \mid \exists n[n \in A \& I(n,a) \in A]\} \cup \{a \in E \mid \exists\alpha\forall n[Sb.(a,\bar{\alpha}(n)) \in A]\}$$

The *$\mathscr{A}$-formulae* (in the set variable $X$) are the class of formulae built up from formulae of the form $(\sigma = \pi)$, $\neg(\sigma = \pi)$, $\sigma \in X$ using $\vee, \&, \exists x, \forall x, \mathscr{A}x, \mathscr{A}^0 x$ where $\sigma, \pi$ are representing terms and $\mathscr{A}x\phi(x)$, $\mathscr{A}^0 x\phi(x)$ are $\exists v\forall x\phi(\bar{v}(x))$, $\forall v\exists x\,\phi(\bar{v}(x))$ respectively. Note that $X$ occurs only positively in an $\mathscr{A}$-formula. An $\mathscr{A}$-formula $\phi(X,x)$ with only one free number variable $x$ *defines* an inductive definition $\Gamma$ if

$$\Gamma(A) = \{n \in \omega \mid \phi(\boldsymbol{A}, \boldsymbol{n}) \text{ is true}\},$$

and such $\Gamma$ will be called *$\mathscr{A}$-inductive definitions*. Note that an $\mathscr{A}$-inductive definition is automatically monotone.

LEMMA 1.2. *Every monotone $\Sigma_1^1$ inductive definition is an $\mathscr{A}$-inductive definition.*

PROOF. Let $R$ be $\Sigma_1^1$ such that

$$x \in \Gamma(\{n \mid \alpha(n) = 0\}) \Leftrightarrow R(\alpha, x)$$

If $\Gamma$ monotone then

$$x \in \Gamma(A) \Leftrightarrow \exists\alpha\forall n[(\alpha(n) = 0 \rightarrow n \in A) \,\&\, R(\alpha, x)]$$

There is a recursive $S$ such that $R$ may be put in the form

$$R(\alpha, x) \Leftrightarrow \exists\beta\forall n S(\bar{\alpha}(m), \bar{\beta}(m), x)$$

So

$$x \in \Gamma(A) \Leftrightarrow \exists\alpha\exists\beta\forall n[(n > 0 \,\&\, \alpha(n-1) = 0 \rightarrow n - 1 \in A) \,\&\, S(\bar{\alpha}(n), \bar{\beta}(n), x)]$$

Let
$$f_0(s) = \langle \pi((s)_0), \cdots, \pi((s)_{lh(s)+1}) \rangle$$
$$f_1(s) = \langle \delta((s)_0), \cdots, \delta((s)_{lh(s)+1}) \rangle \qquad \text{if } s > 0$$
and
$$f_0(0) = f_1(0) = 0.$$
Let
$$g_0(s) = \pi((s)_{lh(s)+1})$$
and
$$g_1(s) = lh(s) + 1.$$

If $\theta(v_0, v_1, v_2)$ is an $\mathscr{A}$-formula defining $S$ then the following $\mathscr{A}$-formula defines $\Gamma$

$$\exists v\forall n[((\bar{v}(n) = 0) \vee \neg(\boldsymbol{g}_0(\bar{v}(n)) = 0) \vee \boldsymbol{g}_1(\bar{v}(n)) \in X) \,\&\, \theta(\boldsymbol{f}_0(\bar{v}(n)), \boldsymbol{f}_1(\bar{v}(n)), z)]$$

LEMMA 1.3. *Any set defined by an $\mathscr{A}$-inductive definition is weakly $\mathscr{A}$-representable.*

PROOF. Let $\phi(X,x)$ be an $\mathscr{A}$ formula defining the inductive definition $\Gamma$. Let $\Phi(X)$ be $\forall v[\phi(X,v)\rightarrow v\in X]$ and let $\phi(x)$ be $\forall X[\Phi(X)\rightarrow x\in X]$ Then

$$n\in\Gamma^{\infty}\Leftrightarrow\phi(\boldsymbol{n}) \text{ is true.}$$

We shall show that $\phi(x)$ weakly $\mathscr{A}$-represents $\Gamma^{\infty}$. Let $T=\{n\in\omega\mid\vdash_{\mathscr{A}}\phi(\boldsymbol{n})\}$. We must show that $\Gamma^{\infty}=T$. Clearly $T\subseteq\Gamma^{\infty}_{;}$ as every $\mathscr{A}$-provable sentence is true. To show $\Gamma^{\infty}\subseteq T$ it is sufficient to show that $\Gamma(T)\subseteq T$. i.e. that $\phi(\boldsymbol{T},x)$ is true $\Rightarrow\vdash_{\mathscr{A}}\Phi(X)\rightarrow\boldsymbol{x}\in X$, By definition of $\Phi(X)$ it is sufficient to show that $\phi(\boldsymbol{T},\boldsymbol{x})$ is true $\Rightarrow\vdash_{\mathscr{A}}\Phi(X)\rightarrow\phi(X,x)$, which is a trivial consequence of the following lemma.

LEMMA 1.4. *For every $\mathscr{A}$-formula $\theta(X)$ with no free number variables*

$$\theta(\boldsymbol{T}) \text{ is true}\Rightarrow\vdash_{\mathscr{A}}\Phi(X)\rightarrow\theta(X).$$

PROOF. By induction on the structure of the $\mathscr{A}$-formula $\theta(X)$. If $\theta(X)$ has the form $(\sigma=\pi)$ or $\neg(\sigma=\pi)$ then the lemma follows from the fact that $\sigma,\pi$ are closed representing terms. If $\theta(X)$ has the form $(\sigma\in X)$ then, if $\theta(\boldsymbol{T})$ is true then there is an $n\in T$ such that $(\sigma=\boldsymbol{n})$ is true, so that $\vdash_{\mathscr{A}}(\sigma=\boldsymbol{n})\,\&\,[\Phi(X)\rightarrow\boldsymbol{n}\in X]$ which implies that $\vdash_{\mathscr{A}}\Phi(X)\rightarrow\theta(X)$.

If $\theta(X)$ has the form $\theta_1(X)\vee\theta_2(X)$ then if $\theta(\boldsymbol{T})$ is true then $\theta_1(\boldsymbol{T})$ is true or $\theta_2(\boldsymbol{T})$ is true. Hence by the induction hypothesis $\vdash_{\mathscr{A}}\Phi(X)\rightarrow\theta_1(X)$ or $\vdash_{\mathscr{A}}\Phi(X)\rightarrow\theta_2(X)$ giving $\vdash_{\mathscr{A}}\Phi(X)\rightarrow\theta(X)$.

If $\theta(X)$ has the form $\theta_1(X)\,\&\,\theta_2(X)$ then the proof is similar to the above.

If $\theta(X)$ has one of the forms $\exists x\phi(X,x),\forall x\phi(X,x),\mathscr{A}x\phi(X,x),\mathscr{A}\check{v}x\phi(X,x)$ then the proof has the same form in each case except that at the appropriate place use s made of the dual $\omega$-rule, $\omega$-rule, $\mathscr{A}$-rule and dual $\mathscr{A}$-rule respectively. We illustrate when $\theta(X)$ has the form $\mathscr{A}x\theta_1(X,x)$. If $\theta(\boldsymbol{T})$ is true then $\exists\alpha\forall n[\theta_1(\boldsymbol{T},\bar{\boldsymbol{\alpha}}(\boldsymbol{n}))$ is true$]$. By the induction hypothesis $\exists\alpha\forall n\vdash_{\mathscr{A}}\Phi(X)\rightarrow\theta_1(X,\bar{\boldsymbol{\alpha}}(\boldsymbol{n}))$. Using the $\mathscr{A}$-rule we get

$$\vdash_{\mathscr{A}}\exists v\forall n[\Phi(X)\rightarrow\theta_1(X,\bar{v}(n))] \text{ and hence } \vdash_{\mathscr{A}}\Phi(X)\rightarrow\theta(X).$$

We summarise the above results in Theorem 1 below. If $\mathscr{C}$ is a class of inductive definitions let Ind($\mathscr{C}$) be the family of those sets 1–1 reducible to sets defined by an inductive definition in $\mathscr{C}$. Let $\Sigma^1_1,\mathscr{A}$ denote the family of monotone $\Sigma^1_1$ and $\mathscr{A}$-inductive definitions respectively.

THEOREM 1. *The following are equivalent for $A\subseteq\omega$*

(1) *$A$ is weakly $\mathscr{A}$-representable.*

(2) $A\in Ind\,(\Sigma^1_1)$

(3) $A\in Ind\,(\mathscr{A})$

PROOF. (1) → (2) If $\theta(v)$ weakly $\mathscr{A}$-represents $A$ then $n \in A \Leftrightarrow \ulcorner\theta(\boldsymbol{n})\urcorner \in \Gamma^\infty$ where $\Gamma$ is the monotone $\Sigma_1^1$ inductive definition given in Lemma 1.1. But $\lambda x \ulcorner\theta(\boldsymbol{x})\urcorner$ is a 1–1 recursive function.

(2) → (3) This follows from Lemma 1.2.

(3) → (1) Let $n \in A \Leftrightarrow f(n) \in \Gamma^\infty$, where $\Gamma$ is an $\mathscr{A}$-inductive definition and $f$ is a 1–1 recursive function. Then by Lemma 1.3 there is a formula $\theta(v)$ such that

$$n \in A \Leftrightarrow \vdash_{\mathscr{A}} \theta(f(\boldsymbol{n})).$$

If $\theta'(v)$ is $\theta(f(v))$ then $\theta'(v)$ weakly $\mathscr{A}$-represents $A$.

§2. **Recursion Theory in a consistent functional.** By a *functional* we shall mean a map of one of the forms $\lambda f x F(f; x)$ or $\lambda f F(f)$ where $f$ ranges over partial functions. We shall always assume that $F$ is *consistent*, where $\lambda f x F(f; x)$ is consistent if $f \subseteq g \Rightarrow \lambda x F(f; x) \subseteq \lambda x F(g; x)$ and $\lambda f F(f)$ is consistent if $f \subseteq g \,\&\, F(f) = y \Rightarrow F(g) = y$ and $f \subseteq g$ means that $f$ is a subfunction of $g$.

We wish to define when a partial function or functional is partial recursive in $\lambda f F(f)$. This will be done by defining an enumeration function $\lambda e f x\{e\}(F, f; x)$ for the functionals partial recursive in $F$ as in [5]. We could use minor modifications of the schemes given there, so that they would apply to functionals $\lambda f F(f)$ that may be defined on partial functions. Rather than do this we will use a simplified set of schemes which will have the same effect. $\langle \lambda x[a](x) \mid a < \omega \rangle$ will denote a standard recursive enumeration of the primitive recursive functions.

Given $\lambda g F(g)$ and $f$, we define $\lambda e x\{e\}(F, f; x)$ to be the least function $q$ satisfying the following schemes:

$$\begin{aligned} q((0,e),x) &= [e](x) \\ q((1,a,b),x) &\simeq q(q(a,x),q(b,x)) \\ q((2,0),x) &\simeq F(\lambda y q(x,y)) \\ q((3,0),x) &\simeq f(x). \end{aligned}$$

As $F$ is consistent it is easy enough to see that the above schemes do determine a least function. We make this precise by giving a monotone inductive definiton $\Gamma$ of $V(F,f) = \{(e,x,y) \mid \{e\}(F,f; x) = y\}$.

Let

$$\Gamma(X) = A_0 \cup A_f \cup \Gamma_1(X) \cup \Gamma_F(X)$$

where

$$\begin{aligned} A_0 &= \{((0,a),x,y) \mid [a](x) = y\} \\ A_f &= \{((3,0),x,y) \mid f(x) = y\} \end{aligned}$$

and

$$\Gamma_1(X) = \{((1,a,b),y,z) \mid \exists y_1 y_2[(a,y,y_1),(b,y,y_2),(y_1,y_2,z) \in X]\}$$

$$\Gamma_F(X) = \{((2,0),y,z) \mid \exists g[F(g) = z \,\&\, \forall uv(g(u) = v \rightarrow (y,u,v) \in X)]\}$$

We now define $V(F,f)$ to be $\Gamma^\infty$. To conclude the definition of the enumeration function we must show that $V(F,f)$ determines the graph of a partial function. i.e. if

$$\mathrm{Funct}(X,Y) \Leftrightarrow \forall exyy'[(e,x,y) \in X \,\&\, (e,x,y') \in Y \rightarrow y = y']$$

we must prove the following lemma.

LEMMA 2.1. *Funct* $(V(F,f), V(F,f))$.

PROOF. This may be proved straightforwardly by proving Funct $(\Gamma^\lambda, \Gamma^\lambda)$ by transfinite induction on the ordinal $\lambda$. But it is convenient to give a proof that does not use ordinals, as we shall need to observe that the proof can be formalised in a system of analysis. So let $X_0 = \{x \in \omega \mid \mathrm{Funct}(\{x\},\Gamma^\infty)\}$. Clearly $\mathrm{Funct}(X_0,\Gamma^\infty)$. By Lemma 2.2 below $\mathrm{Funct}(\Gamma(X_0),\Gamma^\infty)$ as $\Gamma(\Gamma^\infty) = \Gamma^\infty$. i.e. for all $x \in \Gamma(X_0)$ $\mathrm{Funct}(\{x\},\Gamma^\infty)$ which means $\Gamma(X_0) \subseteq X_0$. But this implies that $\Gamma^\infty \subseteq X_0$ which is just another way of stating $\mathrm{Funct}(\Gamma^\infty,\Gamma^\infty)$.

LEMMA 2.2. *For all* $X_0, X_1 \subseteq \omega$, $\mathit{Funct}(X_0,X_1) \Rightarrow \mathit{Funct}(\Gamma(X_0),\Gamma(X_1))$

PROOF. If $\{(e,x) \mid \exists y(e,x,y) \in Y_0\} \cap \{(e,x) \mid \exists y(e,x,y) \in Y_1\} = \varnothing$ ;then Funct $(Y_0, Y_1)$. Hence for all distinct pairs $Y_0, Y_1$ from $A_0$, $A_f, \Gamma_1(X_0), \Gamma_F(X_1)$, we have $\mathrm{Funct}(Y_0, Y_1)$. Trivially, $\mathrm{Funct}(A_0, A_0)$ and Funct $(A_f, A_f)$. Hence to prove the lemma it is sufficient to show that

$$\mathrm{Funct}(X_0,X_1) \Rightarrow \mathrm{Funct}(\Gamma_1(X_0),\Gamma_1(X_1)) \,\&\, \mathrm{Funct}(\Gamma_F(X_0),\Gamma_F(x_1))$$

Now assume $\mathrm{Funct}(X_0,X_1)$

(a) If $((1,a,b),y,z_i) \in \Gamma_1(X_i)$ for $i = 0,1$, then there are $y_1^i, y_2^i$ such that

$$(a,y,y_1^i),\ (b,y,y_2^i),\ (y_1^i,y_2^i,z_i) \in X_i \text{ for } i = 0,1.$$

Hence by hypothesis $y_1^0 = y_1^1$, $y_2^0 = y_2^1$, so that $z_0 = z_1$.

(b) If $((2,0),y,z_i) \in \Gamma_F(X_i)$ for $i = 0, 1$ then there are $g_i$ such that $F(g_i) = z_i$ and

$$g_i(u) = v \Rightarrow (y,u,v) \in X_i \text{ for } i = 0,1.$$

Hence by hypothesis $g_0, g_1$ are compatible and so have a common extension $g$ say. But $F$ is consistent so that $z_0 = F(g_0) = F(g) = F(g_1) = z_1$.

Having proved Lemma 2.1., we may define $\{e\}(F,f;x)$ to be the $y$ such that $(e,x,y) \in V(F,f)$ if there is such a $y$, and undefined otherwise.

We shall write $\{e\}(F,f)$, $\{e\}(F;x)$ for $\{e\}(F,f;0), \{e\}(F,\phi,x)$ respectively, where $\phi$ denotes the completely undefined function. Sometimes we shall omit $F$ if there is no ambiguity.

A functional or function is *partial recursive in F* if it has one of the forms $\lambda fx\{e\}(F,f;x), \lambda f\{e\}(F,f), \lambda x\{e\}(F;x)$ for some $e \in \omega$. The set $A$ of integers is *recursive, semi-recursive in F* if $c_A, \sigma_A$ respectively are partial recursive in $F$ where

$$c_A(x) = \sigma_A(x) = 0 \text{ if } x \in A$$

and

$$c_A(x) = 1,\ \sigma_A(x) \text{ is not defined if } x \notin A.$$

A function or functional is partial recursive in $\lambda f F_0(f), \cdots, \lambda f F_n(f)$ if it is partial recursive in $F$ where

$$F(f) \simeq F_i(\lambda t f(t+1)) \text{ if } f(0) = i \leqq n$$

and is undefined otherwise.

Rather than go through the tedious proof of the equivalence of the notions introduced here with those in the literature we shall list some properties of our notion that would be needed in any such proof. Anyone familiar with Kleene's papers should find no great difficulty in proving these.

PROPOSITION A. Let $q$ be a binary partial function such that there are primitive recursive functions $f_0, f_1$ such that

$$q(f_0(a), x) = [a](x)$$
$$q(f_1(a,b), x) \simeq q(q(a,x), q(b,x)).$$

Let $\mathcal{R}_q = \{\lambda x_1 \cdots x_n q(e,(x_1,\cdots,x_n)) \mid n, e < \omega\}$.
Then $\mathcal{R}_q$ contains all the partial recursive functions, $q \in \mathcal{R}_q$, $\mathcal{R}_q$ is closed under explicit definitions, definitions by cases, primitive recursion and minimalisation. Also the iteration and second recursion theorem hold, i.e. there is a primitive recursive $S$ such that

$$q(S(e,x), y) \simeq q(e,(x,y))$$

and for every $f \in \mathcal{R}_q$ there is an $e$ such that

$$q(e,x) \simeq f(e,x).$$

PROPOSITION B.

(1) *Substitution.* If $f, \lambda gxF'(g;x)$ are partial recursive in $F$ then so is $\lambda xF'(f;x)$.

(2) *Transitivity.* If $F''$ is partial recursive in $F'$ and $F'$ is partial recursive in $F$ then so is $F''$.

(3) *First recursion theorem.* If $\lambda f x F'(f; x)$ is partial recursive in $F$ then the equation

$$F'(f; x) \simeq f(x)$$

has a least solution $f$ partial recursive in $F$.

An important property that may hold is given by the following: $F$ has the *Selection Operator Property* if there is a function $\nu$ partial recursive in $F$ such that $\nu(e)$ is defined $\Leftrightarrow \exists x\{e\}(F; x) = 0$ and $\exists x\{e\}(F; x) = 0 \Rightarrow \{e\}(F; \nu(e)) = 0$.

If $\mathscr{F} \subseteq P(\omega)$ such that $A \supseteq B \in \mathscr{F} \Rightarrow A \in \mathscr{F}$ then we may define a functional $F^{\#}$ by

$$F^{\#}_{\mathscr{F}}(f) \simeq \begin{cases} 0 \text{ if } \{x \mid f(x) = 0\} \in \mathscr{F}^0 \\ 1 \text{ if } \{x \mid f(x) > 0\} \in \mathscr{F} \end{cases}$$

where $\mathscr{F}^0 = \{\omega - A \mid A \notin \mathscr{F}\}$.

Let $F_{\mathscr{F}} = F^{\#}_{\mathscr{F}} \restriction \omega_{\omega}$.

Examples are $E = F_{\{\omega\}}$ and $E_1 = F_{\mathscr{A}}$ where $\mathscr{A} = \{A \subseteq \omega \mid \exists \alpha \forall n \bar{\alpha}(n) \in A\}$.

Clearly every $F_{\mathscr{F}}$ is total. i.e. dom $F_{\mathscr{F}} = \omega_{\omega}$.

In general $F$ need not have the selection operator property. e.g. if $F$ is total with constant value 0 then $F$ does not have the selection operator property.

On the other hand we have:

PROPOSITION C. (Gandy, Hinman) If $E$ is partial recursive in $F$ and either $F$ is total or $F = F^{\#}_{\mathscr{F}}$ for some $\mathscr{F}$ then $F$ has the selection operator property.

For total $F$ the result was announced by Gandy in [2]. For $F$ of the form $F^{\#}_{\mathscr{F}}$ the result is proved by Hinman in [4].

Several important properties follow from the existence of a selection operator.

PROPOSITION D. If $F$ has the selection operator property then

(1) If $A$, $B$ are semi-recursive in $F$ then so are $A \cup B$, $\{y \mid \exists x(x, y) \in A\}$.

(2) $A$ is recursive in $F \Leftrightarrow A, \omega - A$ are semi-recursive in $F$.

(3) $f$ is partial recursive in $F \Leftrightarrow$ the graph of $f$ is semi-recursive in $F$.

§3. **Recursion in $E_1^{\#}$.** In this section we show that the sets weakly (strongly) $\mathscr{A}$-representable are exactly the sets semi-recursive (recursive) in $E_1^{\#}$.

LEMMA 3.1. $\Gamma(X) = A_0 \cup \Gamma_1(X) \cup \Gamma_{E_1^{\#}}(X)$ *is an $\mathscr{A}$-inductive definition,*

PROOF. Clearly $A_0 \cup \Gamma_1(X)$ is definable by an $\mathscr{A}$-formula while

$x \in \Gamma_{E_1^\#}(X) \Leftrightarrow$

$$\exists y\, z\, [x = ((2,0), y, z) \;\&$$

$$[(z = 0 \;\&\; \forall\alpha\exists n(y, \bar{\alpha}(n), 0) \in X) \quad \text{or} \quad (z = 1 \;\&\; \exists\alpha\forall n\exists m(m > 0 \;\&\; (y, \bar{\alpha}(n), m) \in X))]]$$

may also be written as an $\mathscr{A}$-formula.

LEMMA 3.2. *If $\Gamma$ is a $\Sigma_1^1$ monotone inductive definition then $\Gamma^\infty$ is semi-recursive in $E_1^\#$.*

PROOF. By the proof of Lemma 1.2. we may write

$$x \in \Gamma(X) \Leftrightarrow \exists\alpha\forall n[(R_1(\bar{\alpha}(n)) \vee g_1(\bar{\alpha}(n)) \in X) \;\&\; S_1(\bar{\alpha}(n), x)]$$

where $R_1, S_1, g_1$ are recursive. Let $F(f; x) = 1$ if

$$\exists\alpha\forall n[(R_1(\bar{\alpha}(n)) \vee f(g_1(\bar{\alpha}(n))) = 1) \;\&\; S_1(\bar{\alpha}(n), x)]$$

and $F(f; x)$ is undefined otherwise.
Then $F(f; x) \simeq E_1^\#(\lambda y G(f; (y, x)))$ where

$$G(f; (y,x)) \simeq \begin{cases} 1 \text{ if } (R_1(y) \text{ or } f(g_1(y)) = 1) \;\&\; S_1(y,x) \\ 0 \text{ if } (\neg R_1(y) \;\&\; f(g_1(y)) \neq 1) \text{ or } \neg S_1(y,x) \end{cases}$$

$G$ is a partial recursive functional so that $F$ is partial recursive in $E_1^\#$. Let $g$ be the least solution of $F(g; x) \simeq g(x)$. Then $g$ is partial recursive in $E_1^\#$ by the first recursion theorem. Hence $\Gamma^\infty$ is semi-recursive in $E_1^\#$ as $x \in \Gamma^\infty \Leftrightarrow g(x) = 1$.

THEOREM 2.

(a) *$A$ is semi-recursive in $E_1^\# \Leftrightarrow A$ is weakly $\mathscr{A}$-representable.*

(b) *$A$ is recursive in $E_1^\# \Leftrightarrow A$ is strongly $\mathscr{A}$-representable.*

PROOF. (a) If $A$ is semi-recursive in $E_1^\#$ then there is an $e_0$ such that $x \in A \Leftrightarrow \{e_0\}(E_1^\#; x) = 0 \Leftrightarrow (e_0, x, 0) \in \Gamma^\infty$ where $\Gamma$ is defined in Lemma 3.1. Hence by Lemma 3.1. $A \in \mathrm{Ind}(\mathscr{A})$ and by Theorem 1 $A$ is weakly $\mathscr{A}$-representable. Conversely, if $A$ is weakly $\mathscr{A}$-representable then by Theorem 1 $A$ is 1–1 reducible to a set defined by a $\Sigma_1^1$ inductive definition. Hence by Lemma 3.2 $A$ is semi-recursive in $E_1^\#$.

(b) Let $\theta(v)$ be the formula weakly $\mathscr{A}$-representing $V(E_1^\#, \phi)$ as given by applying Lemma 1.3 to the $\mathscr{A}$-inductive definition given by 3.1. By formalising the special case of Lemma 2.1:

$$\mathrm{Funct}(V(E_1^\#, \phi), V(E_1^\#, \phi))$$

we may show that

(1) $$\vdash_{\mathscr{A}} \forall e\, y\, z\, z'[\theta((e,y,z)) \,\&\, \theta((e,y,z')) \to z = z']$$

Now let $A$ be recursive in $E_1^{\#}$. Then there is an $e_0$ such that

$$\forall x[\{e_0\}(E_1^{\#}; x) \leqq 1 \,\&(x \in A \Leftrightarrow \{e_0\}(E_1^{\#}; x) = 0)].$$

Hence

$$x \in A \Leftrightarrow \vdash_{\mathscr{A}} \theta(e_0, \boldsymbol{x}, 0)$$

$$x \notin A \Leftrightarrow \vdash_{\mathscr{A}} \theta(e_0, \boldsymbol{x}, 1)$$

But by (1) $\vdash_{\mathscr{A}} \theta(e_0, \boldsymbol{x}, 1) \Rightarrow \vdash_{\mathscr{A}} \neg\theta(e_0, \boldsymbol{x}, 0)$. Hence $x \notin A \Leftrightarrow \vdash_{\mathscr{A}} \theta(e_0, x, 0)$ so that if $\theta'(v)$ is $\theta(e_0, v, 0)$ then $\theta'(v)$ strongly $\mathscr{A}$-represents $A$.

Conversely if $A$ is strongly $\mathscr{A}$-representable then $A$ and $\omega - A$ are weakly $\mathscr{A}$-representable so that by (a) above $A$ and $\omega - A$ are semi-recursive in $E_1^{\#}$. As $E_1^{\#} = F_{\mathscr{A}}^{\#}$ and $E$ is recursive in $E_1^{\#}$ we may apply proposition $D$ to show that $A$ is recursive in $E_1^{\#}$.

§4. **The Extent of** $\mathrm{Ind}(\Sigma_1^1)$. We have given characterisations of the class of weakly $\mathscr{A}$-representable sets as $\mathrm{Ind}(\Sigma_1^1)$ and as the class of sets semi-recursive in $E_1^{\#}$, but we have not yet indicated the extent of this class. The only interesting upper bound on $\mathrm{Ind}(\Sigma_1^1)$ that we know of is that $\mathrm{Ind}(\Sigma_1^1)$ is a proper subset of the class of $\Delta_2^1$ sets. That $\mathrm{Ind}(\Sigma_1^1) \subseteq \Sigma_2^1$ follows from Theorem 3 of [1] where it is shown that the $\mathscr{A}$-theorems form a $\Sigma_2^1$ set. For monotone $\Sigma_1^1$ $\Gamma$ we have

$$\Gamma^{\infty} = \cap\{A : \Gamma(A) \subseteq A\} = \{n \in \omega \mid \forall\alpha[\forall x(R(\alpha, x) = 0 \Rightarrow \alpha(x) = 0) \Rightarrow \alpha(n) = 0]\}$$

is $\Pi_2^1$ if $R(\alpha, x) \Leftrightarrow x \in \Gamma(\{n \mid \alpha(n) = 0)$ is $\Sigma_1^1$.

Hence $\mathrm{Ind}(\Sigma_1^1) \subseteq \Pi_2^1$ so that $\mathrm{Ind}(\Sigma_1^1) \subseteq \Delta_2^1$. To show that the inclusion is proper observe that if $T$ is the set of $\mathscr{A}$-theorems then $\omega - T \in \Delta_2^1$ while $\omega - T \notin \mathrm{Ind}(\Sigma_1)$ as $T$ is complete for $\mathrm{Ind}(\Sigma_1^1)$.

At first glance it might appear that $E_1$ and $E_1^{\#}$ are of comparable strength so that one might conjecture that $\mathrm{Ind}(\Sigma_1^1)$ is the family of sets semi-recursive in $E_1$. We shall see that this is very far from being the case. This is in contrast to the situation for $E$ and $E^{\#} = F_{\{\omega\}}^{\#}$. Here the sets semi-recursive in $E$ and $E^{\#}$ coincide.

The superjump, introduced by Gandy in [2] is the analogue of the ordinary jump at one type up. We shall here formulate it as a mapping $\mathfrak{S}$ from total functionals $\lambda\alpha F(\alpha)$ to total functionals $\lambda\alpha\mathfrak{S}(F; \alpha)$

$$\mathfrak{S}(F; \alpha) \simeq \begin{cases} \{\alpha(0)\}(F, \lambda t\alpha(t+2); \alpha(1)) + 1 & \text{if this is defined} \\ 0 & \text{otherwise.} \end{cases}$$

Let $F \leqq_T F'$ if $F$ is partial recursive in $F'$ and let $F <_T F'$ if $[F \leqq_T F' \,\&\, F' \not\leqq_T F]$ while $F \equiv_T F'$ if $[F \leqq_T F' \,\&\, F' \leqq_T F]$.

The basic properties of $\mathfrak{S}$ may be formulated as follows:

(1) For total $F, F <_T \mathfrak{S}(F)$ and if $F, F'$ are total

$$F \leqq_T F' \Rightarrow \mathfrak{S}(F) \leqq_T \mathfrak{S}(F').$$

(2) If ${}^2\mathbf{0}$ is the total functional that is everywhere 0 then

$$\mathfrak{S}({}^2o) \equiv_T \mathfrak{S}(E) \equiv_T E_1.$$

THEOREM 3 (Gandy) *$\mathfrak{S}(F)$ is partial recursive in $F$, $E_1^\#$ uniformly in total functionals $F$.*

PROOF. The basic idea of the proof is that for total $F$, $\{e\}(F,\alpha; x)$ is defined if and only if there is no infinite branch in its computation tree. The existence, or non-existence of such a branch can be decided by suitably applying $E_1^\#$. The computation tree for $\{e\}(F,\alpha; x)$ may be described as follows: At the root of the tree place the integer $(e,x)$. If an integer $n$ occurs at a point $P$ of the tree then integers are placed at points immediately above $P$ on the tree according to the following criteria:

If $n = ((0,a),y)$, or $n = ((3,0),y)$ then no integer occurs above $P$ in the tree. If $n = ((1,a,b),y)$ then $(a,y)$, $(b,y)$ and also $(\{a\}(F,\alpha; y), \{b\}(F,\alpha; y))$ if defined occur at points immediately above $P$ in the tree. If the third integer is not defined then call $P$ a *critical* point of the tree. If $n = ((2,0),y)$ then for each $m$ the integer $(y,m)$ occurs at a point immediately above $P$. If $n$ is not of one of the above forms then just the integer $n$ occurs at a point immediately above $P$. By examining the inductive definition of $V(F,\alpha)$ it is not hard to see that $\{e\}(F,\alpha; x)$ is defined iff there is no infinite branch in its computation tree and that if there is no infinite branch no point of the tree can be critical. We shall code finite or infinite branches by the sequence of integers occurring on them. We shall define a functional $G(\alpha; (e,x,y))$ partial recursive in $F$ such that

(1) $G(\alpha; (e,x,\bar{\beta}(n))) = 0$ iff $\beta(0),\cdots,\beta(n-1)$ code a branch on the computation tree.

(2) $G(\alpha; (a,x,\bar{\beta}(n))) = 1$ iff $\beta(0),\cdots,\beta(n-2)$ code a branch $P_0 \cdots P_{n-2}$ on the computation tree, $P_{n-2}$ is not critical but $\beta(n-1)$ does not occur at any point immediately above $P_{n-2}$. Then $E_1^\#(\lambda y\, G(\alpha; (e,x,y)))$ is always defined and $E_1^\#(\lambda y\, G(\alpha; (e,x,y)) = 0$ if $\{e\}(F,\alpha; x)$ is defined. So that by a definition by cases we may prove the theorem.

Before defining $G$ we define a functional $H(\alpha; (a,b))$ partial recursive in $F$ such that

(3) $H(\alpha; ((e,x),b)) = 1$ iff $b$ occurs on the computation tree for $\{e\}(F,\alpha; x)$ immediately above the root point

(4) $H(\alpha; ((e,x),b)) = 0$ iff the root is not a critical point of the tree but $b$ does not occur immediately above this point.

Let

$$H(\alpha; (x,y)) \simeq \begin{cases} 1 & \text{if } S(x,y) \\ H'(\alpha; (x,y)) & \text{if } \neg S(x,y) \end{cases}$$

where $S$ is a recursive relation defined by:

$$\begin{aligned} S((n,a),x),y) \Leftrightarrow & [n = 1 \,\&\, (y = (\pi(a),x) \text{ or } y = (\delta(a),x))] \\ & \text{or } [n = 2 \,\&\, x = \pi(y)] \\ & \text{or } [n \geqq 3 \,\&\, y = ((n,a),x) \,\&\, (n = 3 \Rightarrow a \neq 0)] \end{aligned}$$

and $H'$ partial recursive in $F$ and is defined by:

$$H'(\alpha; (((n,a),x),y))$$
$$\simeq \begin{cases} 1 \text{ if } n = 1 \,\&\, y = (\{\pi(a)\}(F,\alpha; x), \{\delta(a)\}(F,\alpha; x)) \\ 0 \text{ if } n \neq 1 \text{ or } y \neq (\{\pi(a)\}(F,\alpha; x), \{\delta(a)\}(F,\alpha; x)) \end{cases}$$

Now $G$ may be defined by a course of values recursion from $H$.

$$G(\alpha; (e,x,0)) = 1$$

$$G(\alpha,(e,x,\langle b\rangle)) = \begin{cases} 1 \text{ if } b = (e,x) \\ 0 \text{ if } b \neq (e,x) \end{cases}$$

$$G(\alpha; (e,x,s^*\langle a,b\rangle)) \simeq H(\alpha; (a,b)) \text{ if } G(\alpha; (e,x,s^*\langle a\rangle)) = 1$$

and is undefined otherwise.

Properties (1) and (2) for $G$ follow from (3) and (4).

Using the superjump there is a natural method of simultaneously defining a system of notations $\langle S, <_S\rangle$ for ordinals and a hierarchy $\langle F_a \mid a \in S\rangle$ of total functionals. This definition is analogous to the definition of a hierarchy of hyperanalytic predicates given at the end of [6].

S1. $1 \in S, F_1(\alpha) = 0$. S2. If $y \in S$ then $2^y \in S$ and $y <_S 2^y$ and $F_{2y} = \mathfrak{S}(F_y)$. S3. If for all $n$ $y_n = \{y\}\ (F_u; n) \in S \,\&\, y_n <_S y_{n+1}$ and $y_0 = u$ then $3^u 5^y \in S$ and

$y_n <_S 3^u 5^y$ for all $n$ and $F_{3^u 5^y}(\alpha) = F_{y\alpha(0)}(\lambda t\alpha(t+1))$. S4. If $x <_S y$ and $y <_S z$ then $x <_S z$. S5. $x \in S$ and $x <_S y$ only as required by S1–S4.

By Theorem 3 we may easily show that $S$ and $\{(x,y) \mid x <_S y\}$ are semi-recursive in $E_1^{\#}$ and that for all $a \in S$ $F_a$ is recursive in $E_1^{\#}$. Also we may assign an ordinal $|a|$ to each $a \in S$ by

$$|1| = 0$$

$$|2^y| = |y| + 1$$

$$|3^u S^y| = \sup_{n<\omega} |y_n| \text{ where } y_n \text{ is defined in S.3.}$$

and show that if $|a| \leqq |b|$ then $F_a \leqq_T F_b$ for $a,\ b \in S$. Hence if $|a| < |b|$ then $\mathfrak{S}(F_a) \leqq_T F_b$ for $a, b \in S$.

Thus $E_1$ is already recursive in any $F_a$ for $|a| \geqq 2$ while for each $a \in S$ there are functions recursive in $E_1^{\#}$ that are not recursive in $F_a$.

The above definitions raise the following problem: Is every total functional $F$, that is recursive in $E_1^{\#}$, recursive in $F_a$ for some $a \in S$ ?

The answer to this question is negative even if $E_1^{\#}$ is replaced by the superjump. But Platek has recently shown that a type 1 function is recursive in the superjump if and only if it is recursive in some $F_a$ for $a \in S$. Hence $\operatorname{Sup}\{|a| : a \in S\}$ is the sup of the order types of well orderings of sets of invegers recursive in the superjump. On the other hand work of W. Ri:hter suggest that $E_1^{\#}$ may be used to get much larger ordinals than this so that $E_1^{\#}$ appears to be even more powerful than the superjump.

§5. **The weak $\mathscr{A}$-system.** In this section we give a characterisation of the sets recursive and semi-recursive in $E_1$ in terms of strong and weak representability in a system $(A_{w-\mathscr{A}})$ obtained from $(A_{\mathscr{A}})$ by replacing closure under the $\mathscr{A}$-rule by closure under the following weakened form of the $\mathscr{A}$-rule:

$T$ is closed under the *w-$\mathscr{A}$ rule* if

$$\forall n[\phi(n) \in T \text{ or } \neg\phi(n) \in T] \,\&\, \exists\alpha\forall n[\phi(\bar{\alpha}(n)) \in T] \Rightarrow \exists v \forall n \phi(\bar{v}(n)) \in T.$$

Let $\vdash_{w-\mathscr{A}}\phi$ denote that $\phi$ is a $w-\mathscr{A}$ theorem. Note that the $w-\mathscr{A}$ theorems are still closed under the $\omega$-rule and hence also under the dual $\mathscr{A}$-rule.

LEMMA 5.1. *The set of $w-\mathscr{A}$ theorems is semi-recursive in $E_1$.*

PROOF. Let $f_0$ be the partial function whose domain is

$$\{\ulcorner\phi\urcorner \mid \vdash_{w-\mathscr{A}}\phi\} \cup \{\ulcorner\phi\urcorner \mid \vdash_{w-\mathscr{A}}\neg\phi\}$$

and such that $f_0(\ulcorner\phi\urcorner) = 0$ if $\vdash_{w-\mathscr{A}}\phi$ and $f_0(\ulcorner\phi\urcorner) = 1$ if $\vdash_{w-\mathscr{A}}\neg\phi$. We shall prove the lemma by showing that $f_0$ is partial recursive in $\boldsymbol{E}_1$.

Let $Ax.\ E, I, Sb.$ be as in the proof of Lemma 1.1. Let $ng$ be a recursive function such that $ng\ulcorner\phi\urcorner) = \ulcorner\neg\phi\urcorner$ for formulae $\phi$.

Then $f_0$ may be characterised as the least function $f$ such that:

$$a \in Ax. \Rightarrow f(a) = 0.$$

$$[a \in E \,\&\, \forall n[f(Sb.(a,n)) = 0 \text{ or } f(Sb(a,\, n)) = 1]$$

$$\&\, \exists\alpha\forall n f(Sb(a,\, \bar{\alpha}(n))) = 0] \Rightarrow f(a) = 0$$

$$f(ng(a)) = 0 \Rightarrow f(a) = 1.$$

$$\exists n[f(n) = f(I(n,a)) = 0] \Rightarrow f(a) = 1.$$

Define $\boldsymbol{F}$ as follows:

$$\boldsymbol{F}(f\,;(0,a)) = 0 \text{ if } a \in Ax.$$

$$\boldsymbol{F}(f\,;(1,a)) = 0 \text{ if } a \in E \,\&\, \boldsymbol{E}_1(\lambda x(1 - f(Sb.(a,x)))) = 1$$

$$\boldsymbol{F}(f\,;(2,a)) = 1 \text{ if } f(ng(a)) = 0.$$

$$\boldsymbol{F}(f\,;(n+3,a)) = 0 \text{ if } f(n) = f(I(n,a)) = 0.$$

$$\boldsymbol{F}(f\,;x) \text{ is undefined otherwise.}$$

Then $\boldsymbol{F}$ is partial recursive in $\boldsymbol{E}_1$ and the above characterisation of $f_0$ may be rephrased as: $f_0$ is the least function $f$ such that

$$[\exists n \boldsymbol{F}(f\,;(n,a)) = i] \Rightarrow f(a) = i.$$

There is a recursive function $h$ such that

$$\{h(e,a)\}(\boldsymbol{E}_1;\, n) \simeq Z(\boldsymbol{F}(\lambda x\{e\}(\boldsymbol{E}_1;\, x));(n,a))$$

where $Z(y) = 0$ for all $y$.

Let $\boldsymbol{H}(f\,;(e,a)) \simeq \boldsymbol{F}(f\,;(v(h(e,a)),\, a))$ where $v$ is a selection operator for recursion in $\boldsymbol{E}_1$ given by §2 Proposition C. Then $\boldsymbol{H}$ is partial recursive in $\boldsymbol{E}_1$.

By the first recursion theorem there is a least function $g$ partial recursive in $\boldsymbol{E}_1$ such that

$$\boldsymbol{H}(\lambda x g(e,x);\ (e,x)) \simeq g(e,x).$$

Then for every $e$ $g_e = \lambda x\, g(e,x)$ is the least solution of

$$\boldsymbol{H}(f\,;(e,x)) \simeq f(x).$$

By the second recursion theorem choose $e_0$ such that

$$g(e_0, x) \simeq \{e_0\}(\boldsymbol{E}_1; x).$$

We shall show that $f_0 = g_{e_0}$, concluding the proof.

If $\boldsymbol{H}(f_0; (e_0, x) = i$ then $\exists n \boldsymbol{F}(f_0; (n, x)) = i$ so that $f_0(x) = i$ by definition of $f_0$. Hence by the characterisation of $g_{e_0}, g_{e_0} \subseteq f_0$.

Conversely, if $\exists n \boldsymbol{F}(g_{e_0}; (n, a)) = i$ then as $g_{e_0} \subseteq f_0$, $\exists n \boldsymbol{F}(f_0; (n, a)) = i$ so that $f_0(a) = i$. Also $\boldsymbol{H}(g_{e_0}; (e_0, a))$ is defined so that by definition of $g_{e_0}$, $g_{e_0}(a)$ is defined. Suppose $g_{e_0}(a) = i$. Then $f_0(a) = i$ so that $j = i$. Thus we have shown that

$$\exists n \boldsymbol{F}(g_{e_0}; (n, a)) = i \Rightarrow g_{e_0}(a) = i.$$

Hence $f_0 \subseteq g_{e_0}$ by definition of $f_0$.

LEMMA 5.2. *$V(\boldsymbol{E}_1, \Phi)$ is weakly $w - \mathscr{A}$ representable.*

PROOF. The above set is $\Gamma^\infty$ where

$$\Gamma(X) = A_0 \cup \Gamma_1(X) \cup \Gamma_{\boldsymbol{E}_1}(X).$$

Let $\phi(X, x)$ be $\phi_1(X, x) \vee \phi_2(X, x) \vee \phi_3(X, x)$.
where

$$\phi_1(X, x) \text{ is } (\exists a, y, z)\, [x = ((0, a), y, z) \,\&\, [a](y) = z]$$

$$\phi_2(X, x) \text{ is } (\exists a, b, y, z, y_1, y_2)\, [(a, y, y_1) \in X \,\&\, (b, y, y_2) \in X$$
$$\&\, (y_1, y_2, z) \in X \,\&\, x = ((1, a, b), y, z)]$$

and $\phi_3(X, x)$ is

$$(\exists y, z)\, [x = ((2, 0), y, z) \,\&\, \forall n \exists m\, (y, n, m) \in X \,\&\, [(z = 0 \,\&\, \forall \alpha \exists n$$
$$(y, \bar{\alpha}(n), 0) \in X) \vee (z = 1 \,\&\, \exists \alpha \forall n \exists m \quad m > 0 \,\&\, (y, \bar{\alpha}(n), m) \in X]]$$

By the definition of $\phi$ it is an $\mathscr{A}$-formula defining $\Gamma$. As in the proof of Lemma 1.3. define $\Phi(X), \phi(x)$ and

$$T = \{x \in \omega \vdash_{w-\mathscr{A}} \phi(\boldsymbol{x})\}.$$

It is sufficient to show that $\Gamma^\infty = T$. $T \subseteq \Gamma^\infty$ as in the proof of Lemma 1.3. $\Gamma^\infty \subseteq T$ follows from the following lemma.

LEMMA 5.3 *$\phi(\boldsymbol{T}, \boldsymbol{x})$ is true $\Rightarrow \vdash_{w-\mathscr{A}} \Phi(X) \to \phi(X, \boldsymbol{x})$*

PROOF. First note that we may formalise the proof of Funct$(V(\boldsymbol{E}_1, \varnothing), V(\boldsymbol{E}_1, \varnothing))$ to show that

$$(1) \qquad \vdash_{w-\mathscr{A}} (\forall e, x, y, y')\, [\phi((e, x, y)) \,\&\, \phi((e, x, y')) \to y = y']$$

If $\phi(T,x)$ is true then $\phi_i(T,x)$ is true for $i = 1,2$ or 3. If $i = 1$ or 2 then it follows from the proof of Lemma 1.4 that

$$\vdash_{w-\mathscr{A}}\Phi(X) \rightarrow \phi_i(X, x)$$

as the $\mathscr{A}$-rule is not required in these cases. Hence it remains to show that

(2) $$\phi_3(T,x) \text{ is true} \Rightarrow \vdash_{w-\mathscr{A}}\Phi(X) \rightarrow \phi_3(X, x).$$

Suppose $\phi_3(T, x)$ is true. Then there are $y$, $z$ such that

(3) $$x = ((2,0), y, z) \,\&\, \forall n \exists m\, (y, n, m) \in T \qquad \text{and}$$

(4a) $$z = 0 \,\&\, \forall \alpha \exists n\, (y, \bar{\alpha}(n), 0) \in T \qquad \text{or}$$

(4b) $$z = 1 \,\&\, \exists \alpha \forall n \exists m \quad m > 0 \,\&\, (y, \bar{\alpha}(n), m) \in T.$$

Hence as in the proof of Lemma 1.4.

(5) $$\vdash_{w-\mathscr{A}}\Phi(X) \rightarrow [x = ((2,0),\, y, z) \,\&\, \forall n \exists m (y, n, m) \in X]$$

and if (4a) holds then

(6a) $$\vdash_{w-\mathscr{A}}\Phi(X) \rightarrow [z = 0 \;\forall v \exists n (y, \bar{v}(n), 0) \in X]$$

If (4b) holds then we can show

(7) $$\exists \alpha \forall n \vdash_{w-\mathscr{A}} \forall X\; \psi(X, \bar{\alpha}(n))$$

where $\psi(X, v)$ is $[\Phi(X) \rightarrow \exists m \quad m > 0 \,\&\, (y, v, m) \in X]$.
To use the $w - \mathscr{A}$ rule we need to show that for all $n$ either

(7a) $$\vdash_{w-\mathscr{A}} \forall X \psi(X, n)$$

or

(7b) $$\vdash_{w-\mathscr{A}} \neg \forall X \psi(X, n).$$

By (3) $\forall n \exists m \vdash_{w-\mathscr{A}} \phi((y,n,m))$. So for each $n$ either

(8a) $$\exists m > 0 \vdash_{w-\mathscr{A}} \phi((y,n,m))$$

or

(8b) $$\vdash_{w-\mathscr{A}} \phi((y,n,0)).$$

(8a) implies (7a) while (8b) and (1) imply

$$\vdash_{w-\mathscr{A}} \forall m > 0 \neg \phi((y, n, m))$$

which implies (7b). Thus for each $n$ (7a) or (7b) holds, so that we may use the $w - \mathscr{A}$ rule to infer from (7)

$$\vdash_{w-\mathscr{A}} \exists v \forall n \forall X [\psi(X, \bar{v}(n))]$$

which, as $z = 1$, implies

(6a) $$\vdash_{w-\mathscr{A}} \Phi(X) \to [z = 1 \& \exists v \forall n \exists m > 0(y, \bar{v}(n), m) \in X].$$

Thus we have shown that (3) $\Rightarrow$ (5), (4a) $\Rightarrow$ (6a) and (4b) $\Rightarrow$ (6b). From these implications we may infer (2), proving the lemma.

The following consequence of Lemmas 5.1 and 5.2. may be proved along the lines of the proof of Theorem 2.

THEOREM 4. *If* $A \subseteq \omega$ *then*

(a) *A is semi-recursive in* $E_1$ *iff A is weakly* $w - \mathscr{A}$ *representable*

(b) *A is recursive in* $E_1$ *iff A is strongly* $w - \mathscr{A}$ *representable.*

§6. **Generalisations.** In this section we shall give definitions and state results generalising the results of the previous sections.

By a *quantifier* we shall here mean a family $\mathscr{F} \subseteq P(\omega)$. Given such a family we may extend the language of second order arithmetic to allow formulae of the form $\mathscr{F}x\phi(x)$. The interpretation in the standard model is extended to the larger class of formulae so that $\mathscr{F}x\phi(x)$ is true iff $\{n \in \omega \mid \phi(\mathbf{n}) \text{ is true}\} \in \mathscr{F}$. The dual quantifier to $\mathscr{F}$ is $\mathscr{F}^0 = \{\omega - A / A \notin \mathscr{F}\}$. We shall use $\mathscr{F}^0\, x\, \phi(x)$ to abreviate $\neg \mathscr{F} x \neg \phi(x)$. The class of *$\mathscr{F}$-formulae* is the class of formulae containing those of the form $(\sigma = \pi), \neg(\sigma, = \pi), \sigma \in X$ built up using $\vee, \&, \exists x, \forall x, \mathscr{F}x, \mathscr{F}^0 x$, where $\sigma, \pi$ are representing terms. The *$\mathscr{F}$-inductive definitions* are those definable by $\mathscr{F}$-formulae. Below we shall always assume that $\mathscr{F}$ is *positive*. i.e. $A \supseteq B \in \mathscr{F}$ implies $A \in \mathscr{F}$. Then $\mathscr{F}^0$ is also positive and the $\mathscr{F}$-inductive definitions are monotone. Ind($\mathscr{F}$) is defined as in §1 where $\mathscr{F}$ is here used to denote the class of $\mathscr{F}$-inductive definitions.

We now define a theory $(A_{\mathscr{F}})$ obtained from the theory of second order arithmetic by extending the schemas to apply to all formulae in the extended language and adding the following axiom scheme and infinitary rules of inference.

$$\forall x[\phi(x) \to \psi(x)] \to [\mathscr{F}x\phi(x) \to \mathscr{F}x\psi(x)].$$

The $\omega$-rule

The $\mathscr{F}$-rule

The $\mathscr{F}^0$-rule.

where a set $T$ of formulae is closed under the $\mathscr{F}$-rule if

(*) $$\{n \in \omega \mid \phi(\mathbf{n}) \in T\} \in \mathscr{F} \Rightarrow \mathscr{F}x\phi(x) \in T.$$

for all formulae $\phi(x)$. The $\mathscr{F}^0$-rule is obtained by replacing $\mathscr{F}$ in the above by $\mathscr{F}^0$. The theorems of the above system $(A_{\mathscr{F}})$ will be called the $\mathscr{F}$-theorems and $\vdash_{\mathscr{F}}\phi$ will denote that $\phi$ is an $\mathscr{F}$-theorem. Note that every sentence that is an $\mathscr{F}$-theorem is true.

The $\mathscr{F}$-rule may be weakened to the *$w-\mathscr{F}$ rule*, where $T$ is closed under the $w-\mathscr{F}$ rule if (*) holds whenever

$$\forall n[\phi(\boldsymbol{n}) \in T \text{ or } \neg\phi(\boldsymbol{n}) \in T].$$

Similarly for the $w-\mathscr{F}^0$ rule. The theory $(A_{w-\mathscr{F}})$ is obtained as is the theory $(A_{\mathscr{F}})$ except that the $w-\mathscr{F}$ and the $w-\mathscr{F}^0$ rules are used rather than the $\mathscr{F}$ and $\mathscr{F}^0$ rules.

Given $\mathscr{F}$ we may define two functionals $\boldsymbol{F}_{\mathscr{F}}$ and $\boldsymbol{F}^{\#}_{\mathscr{F}}$ where $\boldsymbol{F}_{\mathscr{F}}$ is the restriction of $\boldsymbol{F}^{\#}_{\mathscr{F}}$ to total functions and

$$\boldsymbol{F}^{\#}_{\mathscr{F}}(f) \simeq \begin{cases} 0 \text{ if } \{x \mid f(x)=0\} \in \mathscr{F}^0 \\ 1 \text{ if } \{x \mid f(x)>0\} \in \mathscr{F} \end{cases}$$

and is undefined otherwise.

In the following we shall assume that $\boldsymbol{E}$ is partial recursive in both $\boldsymbol{F}_{\mathscr{F}}$ and $\boldsymbol{F}^{\#}_{\mathscr{F}}$ so that §2 Propositions C and D apply to both $\boldsymbol{F}_{\mathscr{F}}$ and $\boldsymbol{F}^{\#}_{\mathscr{F}}$.

THEOREM 5.

(a) *The following are equivalent for $A \subseteq \omega$:*
  (1) $A \in \mathrm{Ind}(\mathscr{F})$
  (2) *$A$ is weakly $\mathscr{F}$-representable*
  (3) *$A$ is semi-recursive in $\boldsymbol{F}^{\#}_{\mathscr{F}}$.*

(b) *The following are equivalent*
  (1) *$A$ is strongly $\mathscr{F}$-representable.*
  (2) *$A$ and $\omega - A$ are weakly $\mathscr{F}$-representable.*
  (3) *$A$ is recursive in $\boldsymbol{F}^{\#}_{\mathscr{F}}$.*

THEOREM 6.

(a) *$A$ is weakly $w-\mathscr{F}$ representable iff $A$ is semi-recursive in $\boldsymbol{F}_{\mathscr{F}}$.*

(b) *$A$ is strongly $w-\mathscr{F}$ representable if $A$ is recursive in $\boldsymbol{F}_{\mathscr{F}}$.*

The proofs of these results involve no new ideas other than those exhibited in previous sections. Note that Theorems 5 and 6 remain true when the axioms of $(A_{\mathscr{F}})$ or $(A_{w-\mathscr{F}})$ are extended by adding a recursive set of true sentences. The previous sections give proofs of the special case when $\mathscr{F} = \mathscr{A} = \{A \subseteq \omega \mid \exists\alpha\forall n\bar{\alpha}(n) \in A\}$ and $[\mathscr{A}x\phi(x) \leftrightarrow \exists\nu\forall n\phi(\bar{\nu}(n))]$ is added as an axiom for each formula $\phi(x)$. An application to a more familiar case is when $\mathscr{F} = \{\omega\}$, and the only infinitary rule is the $\omega$-rule, when $\boldsymbol{F}_{\mathscr{F}}$ is just $\boldsymbol{E}$.

## References

1. H. B. Enderton, *An infinitistic rule of proof*, J. Symbolic Logic, **32** (1967), 447–451.

2. R. Gandy, *General recursive functionals of finite type and hierarchies of functions*, Symposium on Mathematical Logic at Clermont Ferrand, 1962.

3. A. Grzegorczyk, A. Mostowski and C. Ryll-Nardzewski, *The classical and ω-complete arithmetic*, J. Symbolic Logic, **23** (1958), 188–205.

4. P. Hinman, Hierarchies of effective descriptive set theory, Trans. Amer. Math. Soc. **142** (1969), 111–140.

5. S. C. Kleene, *Recursive functionals and quantifiers of finite types. I,* Trans. Amer. Math. Soc. **91** (1959), 1–52.

6. S. C. Kleene, *Recursive functionals and quantifiers of finite types, II,* Trans. Amer. Math. Soc. **108** (1963), 106–142.

7. Y. N. Moschovakis, *Hyperanalytic Predicates*, Trans. Amer. Math. Soc. **129** (1967), 249–282.

Manchester University
England